CHAPTER 5 THE INTEGRAL

AP EXAM

Multiple Choice Questions

Use scratch paper as necessary to solve each problem. Pick the best answer from the choices provided.

1. C If $\int_0^3 f(x)\,dx = 6$ and $\int_0^2 f(x)\,dx = 4$, then $\int_3^2 f(x)\,dx =$

(A) -10
(B) -2
(C) -1
(D) 2
(E) 10

2. C If $\int_1^4 f(x)\,dx = 7$, then $\int_1^4 (2f(x) + 5)\,dx =$

(A) 12
(B) 19
(C) 24
(D) 29
(E) 57

3. C If $F'(x) = \sqrt{1 + x^3}$ and $F(1) = 5$, then $F(3) =$

(A) 1.230
(B) 3.585
(C) 6.230
(D) 8.535
(E) 11.230

4. If $F(x) = \int_2^x \sqrt{t^3 - 1}\,dt$, then $F'(2) =$

(A) 0
(B) $\frac{1}{2\sqrt{7}}$
(C) $\frac{6}{\sqrt{7}}$
(D) $\sqrt{7}$
(E) $12\sqrt{7}$

5. If $F(x) = \int_{3x}^{\sqrt{\pi}} \cos(t^2)\,dt$, then $F'(x) =$

(A) $-1 - \cos(9x^2)$
(B) $3\cos(9x^2)$
(C) $-3\cos(9x^2)$
(D) $3\sin(9x^2)$
(E) $-3\sin(9x^2)$

6. C Use the following table to compute the left-hand Riemann sum for $\int_1^3 F(x)\,dx$ using four subintervals.

x	1	1.75	2	2.5	3
$F(x)$	4	12	6	12	2
$F'(x)$	-1	3	4	2	7

(A) 3
(B) 15
(C) 17
(D) 32
(E) 34

7. C Use the table above to compute $\int_1^3 F'(x)\,dx$.

(A) -2
(B) 2
(C) 6
(D) 8
(E) 15

8. $\int 6x\sin(x^2)\,dx =$

(A) $3x^2\cos(x^2)+C$

(B) $3\cos(x^2)+C$

(C) $-3\cos(x^2)+C$

(D) $6\cos(x^2)+C$

(E) $-12\cos(x^2)+C$

9. Water flows into a tank at $(6t^2+1)$ gallons per minute for $0\le t\le 2$, with t in minutes. If the tank held 32 gallons when $t=2$, how much water, in gallons, was in the tank when $t=1$?

(A) 7

(B) 14

(C) 15

(D) 17

(E) 18

10. $\int 4x\sqrt{x-1}\,dx =$

(A) $\frac{8}{5}(\sqrt{x-1})^5+\frac{8}{3}(\sqrt{x-1})^3+C$

(B) $\frac{8}{5}(\sqrt{x-1})^5+\frac{2}{3}(\sqrt{x-1})^3+C$

(C) $3x^2(x-1)^{1/2}+C$

(D) $2x^2+\frac{3}{2}(x-1)^{3/2}+C$

(E) $4(x-1)^{3/2}+4(x-1)^{1/2}+C$

11. $\int \frac{dx}{4+16x^2} =$

(A) $\ln|4+16x^2|+C$

(B) $\frac{1}{32}\ln|4+16x^2|+C$

(C) $\frac{1}{8}\arctan(2x)+C$

(D) $\frac{1}{4}\arctan(2x)+C$

(E) $\frac{1}{4}\arctan(4x)+C$

12. $\int \frac{dx}{6x+12} =$

(A) $6\ln|6x+12|+C$

(B) $\frac{1}{6}\ln|x+2|+C$

(C) $\ln|x+2|+C$

(D) $\ln|6x+12|+C$

(E) $6\ln|x+2|+C$

13. If $w=2x$, then $\int_0^2 f(2x)\,dx =$

(A) $\int_0^2 f(w)\,dw$

(B) $\frac{1}{2}\int_0^2 f(w)\,dw$

(C) $\frac{1}{2}\int_0^4 f(w)\,dw$

(D) $\int_0^4 f(w)\,dw$

(E) $2\int_0^1 f(w)\,dw$

14. A population triples every 6 months. How long, in months does it take to double?

(A) 3

(B) 4

(C) $\frac{6\ln 2}{\ln 3}$

(D) $6\ln 2-\ln 3$

(E) $6(\ln 2-\ln 3)$

15. C If $g(x)=\int_0^{2x} f(t)\,dt$, use the table below to compute $g'(3)$.

x	0	3	6
$f(x)$	1	5	7
$f'(x)$	9	11	−4

(A) −1

(B) 5

(C) 10

(D) 11

(E) 14

16. $\int \frac{1}{\sqrt{4-9x^2}}\,dx =$

(A) $\ln\sqrt{4-9x^2}+C$

(B) $2\sqrt{4-9x^2}+C$

(C) $\arcsin(3x)+C$

(D) $\frac{1}{3}\arcsin(3x)+C$

(E) $\frac{1}{3}\arcsin(\frac{3x}{2})+C$

17. C The midpoint Riemann Sum to approximate $\int_1^9 x^2\,dx$ using four subintervals is

(A) $1+9+25+49$

(B) $2(1+9+25+49)$

(C) $2(4+16+36+64)$

(D) $4+16+36+64$

(E) $4(25)$

18. C A population P is growing at a continuous rate of 3% per year. Which of the following equations represents this information?

(A) $P = P_0 e^{1.03t}$

(B) $P = P_0 e^{3t}$

(C) $\frac{dP}{dt} = P_0 e^{0.03t}$

(D) $\frac{dP}{dt} = 0.03P$

(E) $\frac{dP}{dt} = 1.03P$

19. A particle travels on the x-axis with velocity given by $v(t) = 6 \sin\left(\frac{t}{2}\right)$. If the particle is at $x = 1$ when $t = 0$, then its position when $t = \pi$ is $x =$

(A) -11

(B) -5

(C) -2

(D) 7

(E) 13

20. $\int 3^{x/2} dx =$

(A) $3^{x/2} + C$

(B) $\sqrt{3}\, 3^x + C$

(C) $\frac{2}{\ln 3} 3^{x/2} + C$

(D) $(2 \ln 3) 3^{x/2} + C$

(E) $\frac{\ln 3}{2} 3^{x/2} + C$

Free Response Questions

Show all of your work and clearly label any functions, graphs, tables, or other objects that you use. On the AP, your work will be scored on the correctness and completeness of your methods, as well as your actual answer. You will usually not be given credit for answers that don't include supporting work.

1. A particle travels on the x-axis so its velocity at time t is given by $v(t) = \frac{1}{2} - \sin t$, for $0 \le t \le 2\pi$.

   (a) For what values of t is the particle moving to the right?

   (b) If the particle starts at $x = 3$, what is the final position of the particle?

   (c) What is the total distance traveled by the particle?

   (d) When $t = \frac{\pi}{4}$, is the speed of the particle increasing or decreasing? Justify your answer.

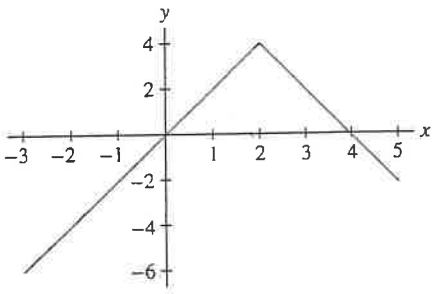
2. Let $g(x) = \int_0^{x^2} \sqrt{t^3 + 64}\, dt$.

   (a) What is the domain of g?

   (b) Find the interval(s) on which g is increasing.

   (c) What is $g''(0)$?

3.



The graph of f above consists of two straight line segments.

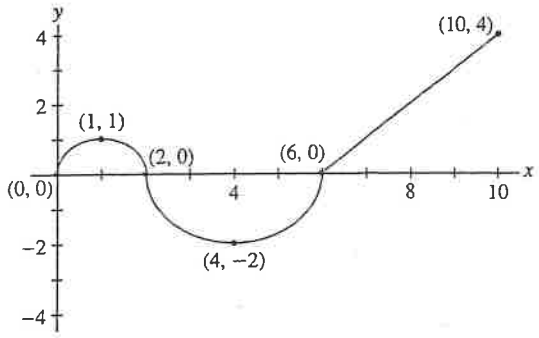
Let $g(x) = \int_2^x f(t)\, dt$ for $-3 \le x \le 5$.

(a) Find values of x for which g has a local maximum. Justify your answer.

(b) What is the maximum value of $g(x)$ on the interval $[-3, 5]$? Justify your answer.

(c) On what subinterval(s) of $(-3, 5)$ is the graph of g concave up? Justify your answer.

4. C



The graph of f given here consists of two semi-circles and one straight line segment. Let $g(x) = \int_1^x f(t)\,dt$, $0 \le x \le 10$.

(a) What is $g(0)$?

(b) For what x, $0 < x < 10$, does $g'(x)$ fail to exist? Explain.

(c) For what x, $0 < x < 10$, does $g''(x)$ fail to exist? Explain.

(d) How many solutions are there to $g(x) = 0$ for $0 \le x \le 10$? Justify your answer.

Answers to odd-numbered questions can be found in the back of the book.

CHAPTER 6
APPLICATIONS OF THE INTEGRAL

Multiple Choice Questions

Use scratch paper as necessary to solve each problem. Pick the best answer from the choices provided.

1. The area between the graphs of $y = x^3$ and $y = 4x$ is
(A) -8
(B) -4
(C) 0
(D) 4
(E) 8

2. The area between the graph of $y = |x|$ and the line $2y - x - 3 = 0$ is
(A) $\frac{3}{4}$
(B) $\frac{6}{4}$
(C) $\frac{9}{4}$
(D) 3
(E) 4

3. The area between the curves $x = y^2$ and $y = x - 2$ is
(A) $\frac{7}{6}$
(B) $\frac{10}{3}$
(C) $\frac{9}{2}$
(D) $\frac{16}{3}$
(E) $\frac{39}{2}$

4. The average value of $f(x) = x^2$ on the interval $[1, 3]$ is
(A) 2
(B) 4
(C) $\frac{13}{3}$
(D) $\frac{26}{3}$
(E) 8

5. C If f is continuous on $[a, b]$, which of the following must be true? There is a c in $[a, b]$ with

I $f(c) = 0$

II $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

III $f(c) = \dfrac{\int_a^b f(x)\,dx}{b - a}$

(A) II only
(B) III only
(C) I and II only
(D) II and III only
(E) I, II, and III

6. A solid has as base the inside of the ellipse $4x^2 + 9y^2 = 36$, and cross sections perpendicular to the x-axis are squares. The total volume of the solid is
(A) 16
(B) 32
(C) 64
(D) 72
(E) 576

7. Fluid is flowing in a tube that has a radius of 3 centimeters. Water is flowing through a circular cross section at a rate of $9 - r^2$ cm/s, where r is the distance from the center of the cross section. What is the total amount (in cubic centimeters) of water that flows through the cross section in 4 seconds?
(A) 18
(B) $\frac{81}{4}$
(C) 72
(D) $\frac{81\pi}{4}$
(E) 162π

8. The region under the graph $y = \sqrt{x}$ and above the x-axis over the interval $4 \le x \le 9$ is rotated about the x-axis. The resulting volume is

(A) $\frac{38}{3}$
(B) $\frac{38\pi}{3}$
(C) $\frac{65}{2}$
(D) $\frac{65\pi}{2}$
(E) $\frac{76\pi}{3}$

9. The region below the graph $y = x^2$ and above the line $y = -1$ over the interval $1 \le x \le 4$ is rotated about the line $y = -1$. The resulting volume is

(A) 21π
(B) 24π
(C) $\frac{1248\pi}{5}$
(D) $\frac{828\pi}{5}$
(E) $\frac{1038\pi}{5}$

10. The region bounded by the y-axis and the graphs $y = 3$ and $y = \sqrt{x}$ is rotated about the y-axis. The resulting volume is

(A) 9π
(B) $\frac{81\pi}{2}$
(C) $\frac{243\pi}{5}$
(D) 81π
(E) 243π

11. C The region bounded by the graphs $y = \sqrt{2x+1}$ and $2y - x = 2$ is rotated about the line $x = 5$. The resulting volume is

(A) 4
(B) 4.266
(C) 12.566
(D) 13.404
(E) 90.933

12. C A solid has base given by the triangle with vertices $(-4, 0)$, $(0, 8)$, and $(4, 0)$. Cross sections perpendicular to the y-axis are semi-circles with diameter in the plane. The volume of the solid is given by

(A) $\displaystyle\int_0^4 \pi(8-2x)^2\,dx$

(B) $\displaystyle\frac{\pi}{2}\left(\int_{-4}^0 (2x+8)^2\,dx + \int_0^4 (8-2x)^2\,dx\right)$

(C) $\displaystyle\int_0^8 \frac{\pi}{8}(8-y)^2\,dy$

(D) $\displaystyle\int_0^8 \frac{\pi}{4}(8-y)^2\,dy$

(E) $\displaystyle\int_0^8 \frac{\pi}{2}(8-y)^2\,dy$

13. C The solid bounded by the graphs of $y = 8 - x^2$ and $y - 3x = 8$ is rotated around the line $x = 4$. The volume is given by

(A) $\pi\displaystyle\int_{-3}^0 (8-x^2)^2 - (8+3x)^2\,dx$

(B) $\pi\displaystyle\int_{-3}^0 \left(8-x^2-(8+3x)\right)^2\,dx$

(C) $\pi\displaystyle\int_{-1}^8 \left(4+\sqrt{8-y}\right)^2 - \left(\frac{y-20}{3}\right)^2\,dy$

(D) $\pi\displaystyle\int_{-1}^8 \left(4-\sqrt{8-y}\right)^2 - \left(\frac{y-20}{3}\right)^2\,dy$

(E) $\pi\displaystyle\int_{-1}^8 \left(4+\sqrt{8-y}-\left(\frac{y-20}{3}\right)\right)^2\,dy$

14. C What integral gives the volume of a solid with base a circle of diameter 6 with center at the origin if cross sections perpendicular to the x-axis are equilateral triangles?

(A) $\displaystyle\int_{-3}^3 \frac{\sqrt{3}}{4}(9-x^2)\,dx$

(B) $\displaystyle\int_0^6 \frac{\sqrt{3}}{4}(9-x^2)\,dx$

(C) $\displaystyle\int_{-6}^6 4\sqrt{3}(36-x^2)\,dx$

(D) $\displaystyle\int_{-3}^3 4\sqrt{3}(9-x^2)\,dx$

(E) $\displaystyle\int_{-3}^3 \sqrt{3}(9-x^2)\,dx$

15. C Circle City has a population density of $\rho(r) = \sqrt{4+r^2}$ for $0 \le r \le 3$, where r is the distance in miles from the center of the city, and $\rho(r)$ is in thousands of people/square mile. Which of the following gives the total population of Circle City?

(A) $\displaystyle\int_0^3 2\pi r\sqrt{4+r^2}\,dr$

(B) $\displaystyle\int_{-3}^3 2\pi r\sqrt{4+r^2}\,dr$

(C) $\displaystyle\int_{-3}^3 \pi(4+r^2)\,dr$

(D) $\displaystyle\int_0^6 2\pi r\sqrt{4+r^2}\,dr$

(E) $\displaystyle\int_{-3}^3 \sqrt{4+r^2}\,dr$

16. A solid cylindrical rod has length 7 inches and radius 0.5 inches. The density of the rod at a point x inches from one end is $2x$ oz/in.^{3}. Note, as the units indicate, that if W is weight, then density is the derivative of W with respect to volume. In ounces, what is the total weight of the rod?

(A) 14
(B) 49
(C) $\frac{49\pi}{4}$
(D) $\frac{49\pi}{2}$
(E) 49π

Use the following table to solve questions 17 and 18.

$x = 0$	$0 < x < 1$	$x = 1$	$1 < x < 2$	$x = 2$
$f(0) = 3$	f is increasing	$f(1) = 10$	f is decreasing	$f(2) = 6$
$f'(0) = 0$	$f'(x) > 0$	$f'(1) = 0$	$f'(x) < 0$	$f'(2) = 0$

17. C Use the table to find the area between the graph of f' and the x-axis for $0 \le x \le 2$. The area is

(A) 0
(B) 3
(C) 4
(D) 11
(E) 19

18. C Use the table to find the average value of f' on $[0, 2]$.

(A) 1.5
(B) 3
(C) 5.5
(D) 6.333
(E) 9.5

19. C A container in the shape of a sphere of radius R contains water whose depth is H. If the origin is placed at the center of the sphere, then the volume of water is given by which of the following integrals? (The density of water is 62.4 lb/ft^3.)

(A) $\displaystyle\int_{-R}^{-H} \pi R^2\, dy$
(B) $\displaystyle\int_{-R}^{-H} \pi(R^2 - y^2)\, dy$
(C) $\displaystyle\int_{0}^{H} \pi(R^2 - y^2)\, dy$
(D) $\displaystyle\int_{-R}^{-H} 62.4\pi(R^2 - y^2)\, dy$
(E) $\displaystyle\int_{-R}^{-R+H} \pi(R^2 - y^2)\, dy$

20. C Let R be the region bounded above by $y = 8 - x^2$ and below by $y = x^2$. What integral gives the volume of the solid obtained by rotating R about the line $y = -1$?

(A) $\displaystyle\int_{-2}^{2} 2\pi x(8 - 2x^2)\, dx$
(B) $\displaystyle\int_{-2}^{2} \pi\left[(9 - x^2)^2 - (1 + x^2)^2\right] dx$
(C) $\displaystyle\int_{-2}^{2} \pi(8 - 2x^2)^2\, dx$
(D) $\displaystyle\int_{0}^{8} 2\pi y(8 - 2y^2)\, dy$
(E) $\displaystyle\int_{0}^{8} \pi(8 - 2y^2)^2\, dy$

Free Response Questions

Show all of your work and clearly label any functions, graphs, tables, or other objects that you use. On the AP, your work will be scored on the correctness and completeness of your methods, as well as your actual answer. You will usually not be given credit for answers that don't include supporting work.

1. A particle travels on the x-axis with acceleration given by $a(t) = 6 - 2t$ for $0 \le t \le 10$, where t is in seconds and distance is in feet. When $t = 0$, the particle is at $x(0) = 2$ with velocity $v(0) = 7$. Include units in your answers.

(a) What is the average acceleration of the particle during these 10 seconds?

(b) What is the average velocity during these 10 seconds?

(c) What is the average speed during these 10 seconds?

2. Let R be the region in the first quadrant bounded above by $y = 4x$ and below by $y = x^3$. Set up, but do not evaluate an integral expression for each of the following.

(a) the area of R

(b) the volume of the solid obtained by rotating R about the y-axis

(c) the volume of the solid obtained by rotating R about the line $y = 20$

3. An empty bowl is in the form of a hemisphere with radius 6 feet.
   (a) Water starts to be pumped into the bowl at the steady rate of 4 cubic feet per minute. How fast is the depth of the water rising when the depth is 2 feet?
   (b) The pumping of the water stops when the depth of the water is 5 feet. What is the total time that water was pumped into the bowl?

4. An old urn is discovered that looks like a cylinder except that the side is wavy. The urn is 6 inches high. Scientists want to estimate the volume of the urn so they measure its circumference at intervals of 2 inches, and produce the following data, measuring to the nearest tenth of an inch.

y	0	2	4	6
$C(y)$	36.1	34.8	32.2	38.6

   (a) Use a left-hand Riemann sum to estimate the volume of the cylinder.
   (b) Someone suggests that the scientists model the volume as being obtained by rotating the curve $x = 6 + 0.4\sin(5y)$ about the y-axis. Set up, but do not evaluate, an integral expression that gives that volume.